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CATEGORICAL PROPERTIES OF PREORDERED INTUITIONISTIC FUZZY APPROXIMATION SPACES

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Abstract. We prove that for any preordered intuitionistic fuzzy approximation space, an intuitionistic fuzzy topology can be created, and conversely, for any intuitionistic fuzzy topology, a reflexive intuitionistic fuzzy relation can be constructed. We also show that there is a relationship, called Galois correspondence, between the functors of these categories. Additionally, by applying certain limitations on the category of intuitionistic fuzzy topological spaces, we obtain an isomorphism between these categories.

1. Introduction

The theory of the rough set was first proposed by Z. Pollack [8]. It is an extension of set theory that deals with incomplete and uncertain information, and it serves as a useful mathematical tool for data inference in the field of intelligent systems research. The fundamental structure of rough set theory is an approximation space, from which upper and lower limit approximations can be derived. These approximations can be used to uncover hidden knowledge in information systems and express it in the form of decision rules [8, 9]. There have been a variety of studies on applications such as fuzzy topology, intuitionistic fuzzy topology, approximation spaces, and intuitionistic approximation spaces, which examine the relationship between fuzzy theory and extended theories using rough sets [3, 4, 1, 5, 7, 12]. However, the approximation space is determined by equivalence relations on the set, which can be a restrictive condition that does not often occur in real-world scenarios, thus we need to weaken this condition in order to develop a more robust theory.

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In the previous works [11, 12, 13], we have examined the relationship between preordered intuitionistic fuzzy approximation spaces and preordered approximation spaces with the weakened preordered relation condition. Specifically, we found that the upper approximation of a set is the set itself if and only if the set is a lower set, whenever the intuitionistic fuzzy relation is reflexive.

In this paper, we aim to further our mathematical understanding of these spaces by investigating their categorical properties. We demonstrate that for any preordered intuitionistic fuzzy approximation space, an intuitionistic fuzzy topology can be constructed, and conversely, for any intuitionistic fuzzy topology, a reflexive intuitionistic fuzzy relation can be established. Additionally, we show that there is a Galois correspondence between the functors of these categories. Furthermore, by imposing certain restrictions on the category of intuitionistic fuzzy topological spaces, we establish an isomorphism between these categories.

2. Preliminaries

Now we list some definitions and properties which we shall use frequently in the following sections.

Let X be a nonempty set. An *intuitionistic fuzzy set* A is an ordered pair

$$A = (\mu_A, \nu_A)$$

where the functions $\mu_A : X \to I$ and $\nu_A : X \to I$ denote the degree of membership and the degree of nonmembership respectively and $\mu_A + \nu_A \leq 1$(see [1]). Obviously, every fuzzy set μ in X is an intuitionistic fuzzy set of the form $(\mu, \tilde{1} - \mu)$.

Throughout this paper, 'IF' stands for 'intuitionistic fuzzy.' IF(X) denotes the family of all intuitionistic fuzzy sets in X, $I \otimes I$ denotes the family of all intuitionistic fuzzy numbers (a, b) such that $a, b \in [0, 1]$ and $a + b \leq 1$, with the order relation defined by

$$(a, b) \leq (c, d) \text{ iff } a \leq c \text{ and } b \geq d.$$

For all $(a, b) \in I \otimes I$, $\widetilde{(a, b)}$ denotes the constant intuitionistic fuzzy set in X such that the membership value is 'a' and the nonmembership value is 'b'.

Definition 2.1. ([2]) An IF set R on $X \times X$ is called an *intuitionistic fuzzy relation* on X. Moreover, R is called

(i) *reflexive* if $R(x, x) = (1, 0)$ for all $x \in X$,

(ii) *symmetric* if $R(x,y) = R(y,x)$ for all $x, y \in X$,
(iii) *transitive* if $R(x,y) \wedge R(y,z) \leq R(x,z)$ for all $x, y, z \in X$,

A reflexive and transitive IF relation is called an IF preorder.

Definition 2.2. ([15]) Let R be an IF relation on X. Then the two functions $\overline{R}, \underline{R} : \mathrm{IF}(X) \to \mathrm{IF}(X)$, defined

$$\overline{R}(A)(x) = \bigvee_{y \in X} (R(x,y) \wedge A(y)),$$
$$\underline{R}(A)(x) = \bigwedge_{y \in X} (R(x,y)^C \vee A(y)),$$

are respectively called the upper approximation operator and the lower approximation operator on X.

Definition 2.3. ([4, 14]) An IF topology $\mathcal{T}$ on X is a family of IF sets in X that is closed under arbitrary suprema and finite infima and contains all constant IF sets. The IF sets in $\mathcal{T}$ are called open, and their complements, closed.

Definition 2.4. ([12]) Let $\mathcal{T}$ be an IF topology on X. Define an IF relation $R_{\mathcal{T}}$ on X by

$$R_{\mathcal{T}}(x,y) = \mathrm{cl}_{\mathcal{T}}\big(y_{(1,0)}\big)(x)$$

for all $(x,y) \in X \times X$. Then $R_{\mathcal{T}}$ is called the *intuitionistic fuzzy relation induced by* $\mathcal{T}$ on X, and $(X, R_{\mathcal{T}})$ is called the *intuitionistic fuzzy approximation space induced by* $\mathcal{T}$ on X.

For each $R \in \mathrm{IF}(X \times X)$,

$$\mathcal{T}_R = \{A \in \mathrm{IF}(X) \mid A = \underline{R}(A)\},$$
$$\Theta_R = \{\underline{R}(A) \mid A \in \mathrm{IF}(X)\},$$

Proposition 2.5. ([13]) Let (X, R) be an IF approximation space. If R is an IF preorder, then

$$\mathcal{T}_R = \Theta_R.$$

Theorem 2.6. ([13]) If (X, R) is a reflective IF approximation space, then $\mathcal{T}_R$ is the Alexandrov IF topology of X

Definition 2.7. ([11]) Let (X, R) be an IF approximation space. Then $A \in \mathrm{IF}(X)$ is called an *intuitionistic fuzzy upper set* in (X, R) if

$$A(x) \wedge R(x,y) \leq A(y), \ \ \forall x, y \in X.$$

Dually, A is called an *intuitionistic fuzzy lower set* in (X, R) if $A(y) \wedge R(x,y) \leq A(x)$ for all $x, y \in X$.

Let R be an IF preorder on X. For $x, y \in X$, the real number $R(x, y)$ can be interpreted as the degree to which '$x \leq y$' holds true. The condition $A(x) \wedge R(x, y) \leq A(y)$ can be interpreted as the statement that if x is in A and $x \leq y$, then y is in A. Particularly, if R is an IF equivalence, then an IF set A is an upper set in (X, R) if and only if it is a lower set in (X, R).

The classical preorder $x \leq y$ can be naturally extended to $R(x, y) = (1, 0)$ in an IF preorder. Obviously, the notion of IF upper sets and IF lower sets agrees with that of upper sets and lower sets in classical preordered space.

THEOREM 2.8. ([12]) Let $\mathcal{T}$ be an IF topology on X which satisfies the axiom

$$\text{(CC)} \quad \mathrm{cl}_{\mathcal{T}}\big((a,b) \wedge A\big) = (a,b) \wedge \mathrm{cl}_{\mathcal{T}}\big(A\big)$$

for any $(a, b) \in I \otimes I$ and $A \in \mathrm{IF}(X)$. Then

(1) $\overline{R_{\mathcal{T}}}$ is a closure operator of $\mathcal{T}$; i.e. $\overline{R_{\mathcal{T}}} = \mathrm{cl}_{\mathcal{T}}$.
(2) $\mathcal{T}$ is Alexandrov.

Other terminologies used in the following sections refer to [11, 12, 13].

3. Relations between Top and PrApp

In this paper, in order to avoid confusion, we consider only the finite universal set, although some results can be extended to an infinite universal set. From now on, we study categorical relationship between the category of intuitionistic fuzzy approximation spaces and the category of intuitionistic fuzzy topologies. Let **App** be the category of all approximation spaces and order-preserving functions, and let **PrApp** be the category of all preordered approximation spaces and order-preserving functions. Let **IFApp** be the category of all intuitionistic fuzzy approximation spaces and order-preserving functions. And let **IFPrApp** be the category of all preordered intuitionistic fuzzy approximation spaces and order-preserving functions. Let **Top** be the category of all topological spaces and continuous functions, and let **IFTop** be the category of all intuitionistic fuzzy topological spaces and continuous functions.

A function $f : (X, \leq_X) \to (Y, \leq_Y)$ between two approximation spaces is called *order-preserving* if $x \leq_X y$ implies $f(x) \leq_Y f(y)$ for all $x, y \in X$. An order-preserving function in IF approximation spaces is defined similarly as follows.

Definition 3.1. A function $f : (X, R_1) \to (Y, R_2)$ between two IF approximation spaces is called *order-preserving* if

$$R_1(x, y) \le R_2(f(x), f(y)) \text{ for all } x, y \in X.$$

Definition 3.2. ([6]) For a topological space (X, τ) and $x, y \in X$, let $x \preccurlyeq y$ if $y \in M$ implies $x \in M$ for each closed set M of X, or equivalently, $x \in \mathrm{cl}_\tau(\{y\})$. Then $\preccurlyeq$ is a preorder on X, and it is called the *specialization order* on X. We denote this preorder by $\Omega(\tau)$.

Remark 3.3. Consider the following in classical topology,

$$\begin{aligned} & y \in F \text{ implies } x \in F \text{ for all closed set } F \subseteq X \\ & \Leftrightarrow x \notin F \text{ implies } y \notin F \text{ for all closed set } F \subseteq X \\ & \Leftrightarrow x \in F^C \text{ implies } y \in F^C \text{ for all closed set } F \subseteq X \\ & \Leftrightarrow x \in U \text{ implies } y \in U \text{ for all open set } U \subseteq X. \end{aligned}$$

Thus, we have the conclusion that the following conditions are equivalent in classical topology:

(1) $y \in F$ implies $x \in F$ for all closed set $F \subseteq X$.
(2) $x \in U$ implies $y \in U$ for all open set $U \subseteq X$.

Proposition 3.4. Define $\Omega : \mathbf{Top} \to \mathbf{PrApp}$ by

$$\Omega(X, \tau) = (X, \Omega(\tau)) \text{ and } \Omega(f) = f,$$

where $\Omega(\tau)$ is the specialization order on X. Then Ω is a functor from **Top** to **PrApp**.

Proof. Suppose that $f : (X, \tau) \to (Y, \upsilon)$ is a continuous function between two topological spaces. Take $x, y \in X$ and $x \preccurlyeq_X y$ with respect to the specialization order on X. Take any open set $B \in \upsilon$. Since f is continuous, $f^{-1}(B)$ is open in (X, τ). Thus $x \in f^{-1}(B)$ implies $y \in f^{-1}(B)$. That is, $f(x) \in B$ implies $f(y) \in B$. Hence $f(x) \preccurlyeq_Y f(y)$ with respect to the specialization order on Y. Therefore $f : \Omega(X, \tau) \to \Omega(Y, \upsilon)$ is order-preserving. □

Suppose that $(X, \preccurlyeq)$ is a preordered approximation space and $A \subseteq X$. Then the family of all upper sets of X is clearly a topology on X, which is called the Alexandrov topology on X and denoted by $\Gamma(\preccurlyeq)$.

Proposition 3.5. Define $\Gamma : \mathbf{PrApp} \to \mathbf{Top}$ by

$$\Gamma(X, \preccurlyeq) = (X, \Gamma(\preccurlyeq)) \text{ and } \Gamma(f) = f.$$

Then Γ is a functor from **PrApp** to **Top**.

Proof. Suppose that $f : (X, \preccurlyeq_X) \to (Y, \preccurlyeq_Y)$ is an order-preserving function between two preordered approximation spaces. Let A be an open set in $\Gamma(Y, \preccurlyeq_Y)$, then A is an upper set in $(Y, \preccurlyeq_Y)$ by definition of $\Gamma(Y, \preccurlyeq_Y)$. We need only check that $f^{-1}(A) \in \Gamma(\preccurlyeq_X)$, i.e. $f^{-1}(A)$ is an upper set in $(X, \preccurlyeq_X)$. Suppose that $x \preccurlyeq_X y$ and $x \in f^{-1}(A)$. Then $f(x) \preccurlyeq_Y f(y)$ and $f(y) \in A$, because f is order-preserving and A is an upper set. Thus $y \in f^{-1}(A)$, and hence $f^{-1}(A)$ is an upper set in $(X, \preccurlyeq_X)$. So $f^{-1}(A) \in \Gamma(\preccurlyeq_X)$. Therefore $f : \Gamma(X, \preccurlyeq_X) \to \Gamma(Y, \preccurlyeq_Y)$ is continuous. Hence Γ is a functor from **PrApp** to **Top**. □

THEOREM 3.6. (Γ, Ω) is a Galois correspondence between the categories **PrApp** and **Top**. Moreover, Ω is a left inverse of Γ, i.e., $\Omega \circ \Gamma(X, \preceq) = (X, \preceq)$ for any preordered approximation space $(X, \preceq)$.

Proof. First of all, we prove that Ω is a left inverse of Γ. Let $(X, \preceq)$ be any preordered approximation space on X, and let $\leq$ be the specialization order on $\Gamma(X, \preceq)$. Then, for any $x, y \in X$,

$$\begin{aligned} & x \preceq y \\ & \Leftrightarrow x \in U \text{ implies } y \in U \text{ for any upper set } U \subseteq X \\ & \Leftrightarrow x \in U \text{ implies } y \in U \text{ for any open set } U \text{ in } \Gamma(X, \preceq) \\ & \Leftrightarrow x \leq y. \end{aligned}$$

Therefore, $\Omega \circ \Gamma(X, \preceq) = (X, \leq) = (X, \preceq)$. Consequently, Ω is a left inverse of Γ.

Secondly, for any $(X, \preceq_X) \in$ **PrApp**, $id_X : (X, \preceq_X) \to \Omega(\Gamma(X, \preceq_X))$ is clearly an order-preserving function. Consider $(Y, \upsilon) \in$ **Top** and an order-preserving function $f : (X, \preceq_X) \to \Omega(Y, \upsilon)$. In order to show that $f : \Gamma(X, \preceq_X) \to (Y, \upsilon)$ is continuous, take $U \in \upsilon$. Suppose that $f^{-1}(U) \notin \Gamma(\preceq_X)$. Then $f^{-1}(U)$ is not an upper set on $(X, \preceq_X)$, i.e., there are x and y such that $x \preceq_X y$, $x \in f^{-1}(U)$ and $y \notin f^{-1}(U)$. So x is not less than or equal to y in $\Omega(\Gamma(X, \preceq_X))$. This is a contradiction, because the order in $(X, \preceq_X)$ and the order in $\Omega(\Gamma(X, \preceq_X))$ are equivalent. Thus $f^{-1}(U) \in \Gamma(\preceq_X)$. Hence $f : \Gamma(X, \preceq_X) \to (Y, \upsilon)$ is continuous. □

4. Relations between IFTop and IFPrApp

From [13], the intuitionistic fuzzy topology $\mathcal{T}_R$ on X induced by a preordered intuitionistic fuzzy approximation space (X, R) is clearly an Alexandrov topology [6] on X and it satisfies the axiom (CC) of Theorem 2.8.

PROPOSITION 4.1. Define $\Phi : \mathbf{IFPrApp} \to \mathbf{IFTop}$ by

$$\Phi(X, R) = (X, \Phi(R)) \text{ and } \Phi(f) = f,$$

where $\Phi(R) = \{A \in \mathrm{IF}(X) \mid A = \underline{R}(A)\}$. Then Φ is a functor from **IFPrApp** to **IFTop**.

Proof. Suppose that $f : (X, R_X) \to (Y, R_Y)$ is an order-preserving function between two preordered IF approximation spaces. It is enough to show that $f : \Phi(X, R_X) \to \Phi(Y, R_Y)$ is continuous. Let $A \in \Phi(R_Y) = \{A \in \mathrm{IF}(Y) \mid A = \underline{R_Y}(A)\}$, then A^C is an upper set in (Y, R_Y^{-1}). Thus

$$A^C\big(f(x)\big) \wedge R_X^{-1}(x, y) \leq A^C\big(f(x)\big) \wedge R_Y^{-1}(f(x), f(y)) \leq A^C\big(f(y)\big).$$

Thus $f^{-1}(A^C) = A^C\big(f\big)$ is an upper set in (X, R_X^{-1}), which means, $A\big(f\big) = f^{-1}(A) \in \Phi(R_X)$. Therefore f is continuous. □

On the other hand, for an intuitionistic fuzzy topological space $(X, \mathcal{T})$, the IF relation is defined by $R_{\mathcal{T}}(x, y) = \mathrm{cl}_{\mathcal{T}}\big(y_{(1,0)}\big)(x)$ from Definition 2.4. We denote it by $\Psi(\mathcal{T})$. Moreover, for an intuitionistic fuzzy topological space $(X, \mathcal{T})$ which satisfies the axiom (CC), the IF relation $\Psi(\mathcal{T})$ on X is clearly an intuitionistic fuzzy preorder on X by Theorem 3.7 of [12].

Let $\mathbf{IFTop}_{\mathrm{CC}}$ be the category of all intuitionistic fuzzy topological spaces which satisfy the axiom (CC) and continuous functions. If we restrict Ψ on $\mathbf{IFTop}_{\mathrm{CC}}$, we have the following:

PROPOSITION 4.2. Define $\Psi : \mathbf{IFTop}_{\mathrm{CC}} \to \mathbf{IFPrApp}$ by

$$\Psi(X, \mathcal{T}) = (X, \Psi(\mathcal{T})) \text{ and } \Psi(f) = f,$$

where $\Psi(\mathcal{T})(x, y) = \mathrm{cl}_{\mathcal{T}}\big(y_{(1,0)}\big)(x)$. Then Ψ is a functor from $\mathbf{IFTop}_{\mathrm{CC}}$ to **IFPrApp**.

Proof. Suppose that $f : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ is a continuous function between two IF topological spaces which satisfy the axiom (CC). We will show that $f : \Psi(X, \mathcal{T}_X) \to \Psi(Y, \mathcal{T}_Y)$ is order-preserving. Take any $x, y \in X$. Since f is continuous, $f(\mathrm{cl}_{\mathcal{T}_X}(A)) \subseteq \mathrm{cl}_{\mathcal{T}_Y}(f(A))$ for any IF set $A \in \mathrm{IF}(X)$. Thus

$$\begin{aligned}
R_{\mathcal{T}_Y}(f(x), f(y)) &= \mathrm{cl}_{\mathcal{T}_Y}\big(f(y)_{(1,0)}\big)(f(x)) \\
&\geq f\big(\mathrm{cl}_{\mathcal{T}_X}\big(y_{(1,0)}\big)\big)(f(x)) \\
&= \sup\{\mathrm{cl}_{\mathcal{T}_X}\big(y_{(1,0)}\big)(t) \mid t \in f^{-1}(f(x))\} \\
&\geq \mathrm{cl}_{\mathcal{T}_X}\big(y_{(1,0)}\big)(x) \\
&= R_{\mathcal{T}_X}(x, y).
\end{aligned}$$

Hence f is order-preserving. □

THEOREM 4.3. (Ψ, Φ) is a Galois correspondence between the categories **IFTop** and **IFPrApp**. Moreover, Ψ is a left inverse of Φ, i.e., $\Psi \circ \Phi(X, R) = (X, R)$ for any preordered IF approximation space on X.

Proof. First of all, we prove that Ψ is a left inverse of Φ. Let (X, R) be any preordered IF approximation space on X. Then, by Theorem 3.3 of [12], $R_{\mathcal{T}_R} = R$. Therefore, $\Psi \circ \Phi(X, R) = (X, R)$. Consequently, Ψ is a left inverse of Φ.

Now, we prove that (Ψ, Φ) is a Galois correspondence. Take any $(X, \mathcal{T}) \in$ **IFTop**. Then $R_{\mathcal{T}}$ is a reflexive IF relation on X induced by $\mathcal{T}$. By Theorem 3.2 of [12], $\underline{R_{\mathcal{T}}}(A) \subseteq \text{int}_{\mathcal{T}}(A) \subseteq A$ for any IF set $A \in \text{IF}(X)$. Let U be an open set in $\Phi \circ \Psi(X, \mathcal{T})$, then $\underline{R_{\mathcal{T}}}(U) = \text{int}_{\mathcal{T}}(U) = U$ by definition of Φ. Thus, U is open in $(X, \mathcal{T})$. Therefore $id_X : (X, \mathcal{T}) \to \Phi \circ \Psi(X, \mathcal{T})$ is continuous. Consider $(Y, R_Y) \in$ **IFPrApp** and a continuous function $f : (X, \mathcal{T}) \to \Phi(Y, R_Y)$. In order to show that $f : \Psi(X, \mathcal{T}) \to (Y, R_Y)$ is order-preserving, take any $x, y \in X$. Since $f : (X, \mathcal{T}) \to \Phi(Y, R_Y)$ is continuous, $f(\text{cl}_{\mathcal{T}}(A)) \subseteq \text{cl}_{\Phi(R_Y)}(f(A))$ for any IF set $A \in \text{IF}(X)$. Thus,

$$\begin{aligned} R_Y(f(x), f(y)) &= \text{cl}_{\Phi(R_Y)}\big(f(y)_{(1,0)}\big)(f(x)) \\ &\geq f\big(\text{cl}_{\mathcal{T}}\big(y_{(1,0)}\big)\big)(f(x)) \\ &\geq \text{cl}_{\mathcal{T}}\big(y_{(1,0)}\big)(x) \\ &= R_{\mathcal{T}}(x, y). \end{aligned}$$

Hence $f : \Psi(X, \mathcal{T}) \to (Y, R_Y)$ is an order-preserving function. □

PROPOSITION 4.4. Suppose that $(X, \mathcal{T})$ is an IF topological space. Then the following conditions are equivalent:

(1) $(X, \mathcal{T}) = \Phi \circ \Psi(X, \mathcal{T})$.
(2) $(X, \mathcal{T})$ satisfies the axiom (CC).

Proof. By Theorem 3.7 in [12], it is obvious. □

THEOREM 4.5. **IFTop**$_{\text{CC}}$ and **IFPrApp** are isomorphic.

Proof. By Theorem 4.3 and Proposition 4.4, it is obvious. □

5. Relations between Top and IFTop

In this section, we define some functors between **IFTop** and **Top**.

PROPOSITION 5.1. Define $\rho : \mathbf{IFTop} \to \mathbf{Top}$ by

$$\rho(X, \mathcal{T}) = (X, \rho(\mathcal{T})) \text{ and } \rho(f) = f,$$

where $\rho(\mathcal{T}) = \{A^{-1}\big((1,0)\big) \mid A \in \mathcal{T} \text{ and } A(x) = (1,0) \text{ or } (0,1) \text{ for all } x \in X\}$. Then ρ is a functor from **IFTop** to **Top**.

Proof. Clearly $\rho(\mathcal{T})$ is a topology on X. Next, we show that if $f : (X, \mathcal{T}) \to (Y, \mathcal{U})$ is continuous, then $f : (X, \rho(\mathcal{T})) \to (Y, \rho(\mathcal{U}))$ is continuous. Let $B \in \rho(\mathcal{U})$. Then $\chi_B \in \mathcal{U}$. Since $f : (X, \mathcal{T}) \to (Y, \mathcal{U})$ is continuous, $f^{-1}\big(\chi_B\big) = \chi_B\big(f\big) \in \mathcal{T}$. In order to show that $f^{-1}(B) \in \rho(\mathcal{T})$, consider

$$\begin{aligned} x \in f^{-1}(B) &\Leftrightarrow f(x) \in B \\ &\Leftrightarrow \chi_B(f(x)) = (1,0) \\ &\Leftrightarrow f^{-1}\big(\chi_B\big)(x) = (1,0). \end{aligned}$$

So $f^{-1}(B)$ is the crisp set equivalent of $f^{-1}\big(\chi_B\big)$. Therefore $f^{-1}(B) \in \rho(\mathcal{T})$, and hence $f : (X, \rho(\mathcal{T})) \to (Y, \rho(\mathcal{U}))$ is continuous. Thus ρ is a functor. $\square$

We will consider that the functor ω embeds the category of topological spaces as a full subcategory in the category of IF topological spaces.

PROPOSITION 5.2. Define $\omega : \mathbf{Top} \to \mathbf{IFTop}$ by

$$\omega(X, \tau) = (X, \omega(\tau)) \text{ and } \omega(f) = f,$$

where $\omega(\tau)$ is an IF topology generated by family of IF characteristic functions and constant IF sets, i.e., generated by $\{\chi_A \mid A \in \tau\} \bigcup \{\widetilde{(a,b)} \mid (a,b) \in I \otimes I\}$. Then ω is a functor from **Top** to **IFTop**.

Proof. Clearly $\omega(\tau)$ is an IF topology. Next, we show that if $f : (X, \tau) \to (Y, \upsilon)$ is continuous, then $f : (X, \omega(\tau)) \to (Y, \omega(\upsilon))$ is continuous. Let $\chi_B \in \omega(\upsilon)$. Then $B \in \upsilon$, so $f^{-1}(B) \in \tau$. Hence $\chi_{f^{-1}(B)} \in \omega(\tau)$. Consider

$$\begin{aligned} \chi_{f^{-1}(B)}(x) = (1,0) &\Leftrightarrow x \in f^{-1}(B) \\ &\Leftrightarrow f(x) \in B \\ &\Leftrightarrow \chi_B(f(x)) = (1,0) \\ &\Leftrightarrow f^{-1}\big(\chi_B\big)(x) = (1,0). \end{aligned}$$

It is shown that $\chi_{f^{-1}(B)} = f^{-1}\big(\chi_B\big)$. Thus $f^{-1}\big(\chi_B\big) \in \omega(\tau)$. Consider $f^{-1}(\widetilde{(a,b)}) = \widetilde{(a,b)} \in \omega(\tau)$. Hence ω is a functor. $\square$

Theorem 5.3. The functor ρ : **IFTop** $\to$ **Top** is a left adjoint of the functor ω : **Top** $\to$ **IFTop**.

Proof. For any $(X, \mathcal{T})$ in **IFTop**, $id_X : (X, \mathcal{T}) \to \omega(\rho(X, \mathcal{T}))$ is a continuous function. Consider $(Y, \upsilon) \in$ **Top** and a continuous function $f : (X, \mathcal{T}) \to \omega(Y, \upsilon)$. In order to show that $f : \rho(X, \mathcal{T}) = (X, \rho(\mathcal{T})) \to (Y, \upsilon)$ is continuous, let $B \in (Y, \upsilon)$. Then $\chi_B \in \omega(\upsilon)$. Since $f : (X, \mathcal{T}) \to \omega(Y, \upsilon)$ is continuous, $f^{-1}\big(\chi_B\big) \in \mathcal{T}$. Since $\chi_{f^{-1}(B)} = f^{-1}\big(\chi_B\big)$, $f^{-1}(B) \in \rho(\mathcal{T})$. Hence $f : \rho(X, \mathcal{T}) = (X, \rho(\mathcal{T})) \to (Y, \upsilon)$ is continuous. Therefore id_X is a ω-universal function for $(X, \mathcal{T})$ in **IFTop**. □

Let $\textbf{IFTop}_{\mathrm{C}}$ be the category of all intuitionistic fuzzy topological spaces whose elements are of the form χ_A for some $A \subseteq X$, and continuous functions.

Theorem 5.4. Two categories **Top** and $\textbf{IFTop}_{\mathrm{C}}$ are isomorphic.

Proof. Define ω_C : **Top** $\to \textbf{IFTop}_{\mathrm{C}}$ by

$$\omega_C(X, \tau) = (X, \omega_C(\tau)) \text{ and } \omega_C(f) = f,$$

where $\omega_C(\tau)$ is the family of all IF characteristic functions of every open set in (X, τ), i.e., $\omega_C(\tau) = \{\chi_A \mid A \in \tau\}$. Consider the restriction $\rho_C : \textbf{IFTop}_{\mathrm{C}} \to$ **Top** of the functor ρ. Then ω_C and ρ_C are functors. Clearly $\rho_C\big(\omega_C(X, \tau)\big) = \rho_C\big(X, \omega_C(\tau)\big) = \big(X, \rho_C(\omega_C(\tau))\big) = (X, \tau)$ for any $(X, \tau) \in$ **Top**. Moreover, for any $(X, \mathcal{T}) \in \textbf{IFTop}_{\mathrm{C}}$, $\omega_C\big(\rho_C(X, \mathcal{T})\big) = (X, \mathcal{T})$. Hence the result follows. □

Theorem 5.5. The category $\textbf{IFTop}_{\mathrm{C}}$ is a reflective full subcategory of **IFTop**.

Proof. Clearly $\textbf{IFTop}_{\mathrm{C}}$ is a full subcategory of **IFTop**. Take any $(X, \mathcal{T})$ in **IFTop**. Define $\mathcal{T}^* = \{\chi_A \mid \chi_A \in \mathcal{T}\}$. Then $(X, \mathcal{T}^*) \in \textbf{IFTop}_{\mathrm{C}}$ and $id_X : (X, \mathcal{T}) \to (X, \mathcal{T}^*)$ is continuous. Consider $(Y, \mathcal{U}) \in \textbf{IFTop}_{\mathrm{C}}$ and a continuous function $f : (X, \mathcal{T}) \to (Y, \mathcal{U})$. We need only to check $f : (X, \mathcal{T}^*) \to (Y, \mathcal{U})$ is a continuous function. Let $\chi_B \in \mathcal{U}$. Since $f : (X, \mathcal{T}) \to (Y, \mathcal{U})$ is continuous, $f^{-1}(\chi_B) \in \mathcal{T}$. By the definition of $\mathcal{T}^*$, $f^{-1}(\chi_B) = \chi_B(f) \in \mathcal{T}^*$. Hence $f : (X, \mathcal{T}^*) \to (Y, \mathcal{U})$ is a continuous function. □

Corollary 5.6. The category **Top** is a reflective full subcategory of **IFTop**.

6. Other relations

Proposition 6.1. Define $\iota : \mathbf{PrApp} \to \mathbf{IFPrApp}$ by

$$\iota(X, \preccurlyeq) = (X, \iota(\preccurlyeq)) \text{ and } \iota(f) = f,$$

where $\iota(\preccurlyeq)(x, y) = (1, 0)$ if $x \preccurlyeq y$; otherwise $\iota(\preccurlyeq)(x, y) = (0, 1)$. Then ι is a functor from **PrApp** to **IFPrApp**.

Proof. Suppose that $f : (X, \preccurlyeq_X) \to (Y, \preccurlyeq_Y)$ is an order-preserving function between two preordered approximation spaces. It is enough to show that $f : \iota(X, \preccurlyeq_X) \to \iota(Y, \preccurlyeq_Y)$ is an order-preserving function. Obviously, $\iota(X, \preccurlyeq_X)$ and $\iota(Y, \preccurlyeq_Y)$ are preordered IF approximation spaces. Let $x, y \in X$ with $\iota(\preccurlyeq_X)(x, y) = (1, 0)$. Then $x \preccurlyeq_X y$ in X. Since $f : (X, \preccurlyeq_X) \to (Y, \preccurlyeq_Y)$ is order-preserving, $f(x) \preccurlyeq_Y f(y)$. So $\iota(\preccurlyeq_Y)\big(f(x), f(y)\big) = (1, 0)$. Thus $f : \iota(X, \preccurlyeq_X) \to \iota(Y, \preccurlyeq_Y)$ is an order-preserving function. □

Undoubtedly, a preordered approximation space $(X, \preccurlyeq)$ can be regarded as a preordered intuitionistic fuzzy approximation space (X, R) such that the range of R is $I \otimes I$. In this way, we have an embedding $\iota : \mathbf{PrApp} \to \mathbf{IFPrApp}$ of the category of preordered approximation spaces into the category of preordered intuitionistic fuzzy approximation spaces as a full subcategory.

For a preordered approximation space $(X, \preccurlyeq)$, let $R = \iota(\preccurlyeq)$. Precisely, $R(x, y) = (1, 0)$ if $x \preccurlyeq y$; otherwise $R(x, y) = (0, 1)$. Then $\Phi(R)$ is the collection of all upper sets on X whose range is $I \otimes I$. We will show that after the following proposition.

Proposition 6.2. Suppose that $(X, \preccurlyeq)$ is a preordered approximation space and $A : X \to I \otimes I$ is a function. Then the following are equivalent:

(1) A is an upper set in $(X, \iota(\preccurlyeq))$.
(2) A^C is a lower set in $(X, \iota(\preccurlyeq))$.
(3) $A : (X, \preccurlyeq) \to (I \otimes I, \leq)$ is order-preserving.

Proof.

$$\begin{aligned} &A^C \text{ is a lower set in } (X, \iota(\preccurlyeq)) \\ &\quad \Leftrightarrow A^C(y) \wedge \iota(\preccurlyeq)(x, y) \leq A^C(x) \text{ for all } x, y \in X \\ &\quad \Leftrightarrow A^C(x) \geq A^C(y) \text{ whenever } x \preccurlyeq y \\ &\quad \Leftrightarrow A(x) \leq A(y) \text{ whenever } x \preccurlyeq y \\ &\quad \Leftrightarrow A : (X, \preccurlyeq) \to (I \otimes I, \leq) \text{ preserves order.} \end{aligned}$$

$$\begin{aligned} A^C &\text{ is a lower set in } (X, \iota(\preccurlyeq)) \\ &\Leftrightarrow A^C(y) \wedge \iota(\preccurlyeq)(x,y) \leq A^C(x) \text{ for all } x, y \in X \\ &\Leftrightarrow A^C(x) \geq A^C(y) \text{ whenever } x \preccurlyeq y \\ &\Leftrightarrow A(x) \leq A(y) \text{ whenever } x \preccurlyeq y \\ &\Leftrightarrow A(x) \wedge \iota(\preccurlyeq)(x,y) \leq A(y) \text{ for all } x, y \in X \\ &\Leftrightarrow A \text{ is an upper set in } (X, \iota(\preccurlyeq)). \end{aligned}$$

□

PROPOSITION 6.3. Let $(X, \preccurlyeq)$ be a preordered approximation space, and put $R = \iota(\preccurlyeq)$. Then

$$\underline{R}(A) = A \text{ if and only if } A \text{ is an upper set in } (X, \iota(\preccurlyeq)).$$

Proof. By Proposition 6.2 and Corollary 3.5 in [10], it is obvious. □

PROPOSITION 6.4. Let $(X, \preccurlyeq)$ be the preordered approximation space, and put $R = \iota(\preccurlyeq)$. Then $\underline{R}(A) = A$ if and only if $A_{(1,0)}$ is an upper set in $(X, \preccurlyeq)$.

Proof.

$$\begin{aligned} \underline{R}(A) = A &\Leftrightarrow A \text{ is an upper set in } (X, \iota(\preccurlyeq)) \\ &\Leftrightarrow A(x) \leq A(y) \text{ whenever } x \preccurlyeq y \\ &\Leftrightarrow x \in \chi_{A_{(1,0)}} \text{ implies } y \in \chi_{A_{(1,0)}} \text{ whenever } x \preccurlyeq y \\ &\Leftrightarrow A_{(1,0)} \text{ is an upper set in } (X, \preccurlyeq). \end{aligned}$$

□

THEOREM 6.5. The following diagram commutes.

$$\begin{array}{ccc} \mathbf{PrApp} & \xrightarrow{\quad\Gamma\quad} & \mathbf{Top} \\ \Big\downarrow \iota & & \Big\downarrow \omega \\ \mathbf{IFPrApp} & \xrightarrow{\quad\Phi\quad} & \mathbf{IFTop} \end{array}$$

Proof. Suppose that $(X, \preccurlyeq)$ is a preordered approximation space. Then $\iota(\preccurlyeq)(x,y) = (1,0)$ if $x \preccurlyeq y$; otherwise $\iota(\preccurlyeq)(x,y) = (0,1)$. So, if we take an IF set A in $\Phi\big(\iota(X, \preccurlyeq)\big)$, then, by the definition of Φ, $\underline{R}(A) = A$ where $R = \iota(\preccurlyeq)$. On the other hand, $\Gamma(\preccurlyeq)$ is the family of all upper sets of $(X, \preccurlyeq)$. And $\omega\big(\Gamma(\preccurlyeq)\big)$ is generated by $\{\chi_U \mid U \text{ is an upper set of } (X, \preccurlyeq)\} \bigcup \{\widetilde{(a,b)} \mid (a,b) \in I \otimes I\}$. So, by Proposition 6.4, $\omega\big(\Gamma(X)\big) = \Phi\big(\iota(X)\big)$. □

EXAMPLE 6.6. Let $X = [0,1]$ and $\preccurlyeq$ an usual order on X. Then $([0,1], \preccurlyeq)$ is a preordered approximation space. Put $\mathcal{R} = \iota(\preccurlyeq)$. Then we have the following;

$$\mathcal{R}(x,y) = \begin{cases} (1,0) & \text{if} \quad x \preccurlyeq y, \\ (0,1) & \text{otherwise.} \end{cases}$$

Now, we will construct $\Phi\big(\mathcal{R}\big)$. First of all, $\widetilde{(a,b)} \in \Phi\big(\mathcal{R}\big)$, because $\underline{\mathcal{R}}\big(\widetilde{(a,b)}\big) = \widetilde{(a,b)}$ if $\mathcal{R}$ is reflexive. And, by Proposition 6.3, $\underline{\mathcal{R}}\big(A\big) = A \Leftrightarrow A$ is an upper set in $([0,1], \mathcal{R})$ for any $A \in \mathrm{IF}(X)$. Since $A(x) \wedge \mathcal{R}(x,y) \leq A(y)$ for all $x, y \in [0,1]$, $A : [0,1] \to [0,1]$ is clearly an increasing function, i.e. an order-preserving function. So $\Phi\big(\mathcal{R}\big) = \{\widetilde{(a,b)} \mid (a,b) \in I \otimes I\} \bigcup \{A \in \mathrm{IF}(X) \mid A \text{ is order-preserving } \}$. On the contrary, $\Gamma(\preccurlyeq)$ is the collection of all upper sets in $([0,1], \preccurlyeq)$. Then $\omega\big(\Gamma(\preccurlyeq)\big)$ is the intuitionistic fuzzy topology generated by $\{\chi_A \mid A \in \Gamma(\preccurlyeq)\} \bigcup \{\widetilde{(a,b)} \mid (a,b) \in I \otimes I\}$. Hence $\Phi\big(\iota(\preccurlyeq)\big) = \omega\big(\Gamma(\preccurlyeq)\big)$.

7. Conclusion

In this paper, we provided a detailed analysis of the categorical relationship between four different spaces that we have studied. We described how the topological space and the approximation space, as well as the intuitionistic fuzzy topological space and the intuitionistic fuzzy approximation space, are closely related to each other. We also showed that there is a Galois correspondence between the functors of these categories. Additionally, by applying certain limitations on the category of intuitionistic fuzzy topological spaces, we obtained an isomorphism between these categories. We hope that the results of this paper will aid in further research in various fields such as mathematics and computer applications.

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